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Proof for Consecutive N Product

Before trying to prove the main claim, it will make it easier to prove three other lemmas (I wrote them each on their own page):

Lemma 1: The function, f(x) = x mod n is closed under addition.

Symbollically: a mod n + b mod n = (a+b) mod n

Proof: Let the following integers have values such that:

p = a div n & s = a mod n

q = b div n & t = b mod n

r = a mod n + b mod n = s + t

Then it is sufficient to show that: r = (a+b) mod n

By the definition of the modulo operation, a = pn +s & b = qn + t.

a+b = pn + s + qn + t = (p+q)n + s + t = (p+q)n + r

Again by the definition of the modulo operation, we get:

(a+b) mod n = r

QED

Lemma 2: Consecutive natural numbers have unique remainders when divided by n > 1.

Symbolically: For all x in N and n > 1, (x+1) mod n != x mod n

Proof: By the way of contradiction, suppose not. That is assume ~[For all x in N and n > 1, (x+1) mod n != x mod n] or equivalently,

There exists an x in N and n >1 such that (x+1) mod n = x mod n

Using mini claim1, we can rewrite the equation as:

x mod n + 1 mod n = x mod n

Subtracting x mod n from both sides gives us:

1 mod n = 0

This is only true when n = 1, but this contradicts the fact that n > 1. Thus we conclude that our assumption was mistaken and the original claim is true.

QED

Lemma 3: A product of n consecutive natural numbers is divisible by n.

Symbollically, For all n in N,



Let n be a natural number and .



Then it is sufficient to show that n|a is true.

By the definition of mod, we know that given an integer x, the maximum and minimum values of x mod n are n-1 and 0, respectively. Thus, there are n possible values of x mod n.

Additionally, given n consecutive integers, each of their corresponding remainders are unique by lemma 2.

Using these two facts, we can determine that only one of the numbers in the set of n consecutive natural numbers (call it x) has x mod n = 0. This means n|x is true. If we multiply the rest of the numbers to x so that a = x\*(rest of the numbers), by definition of divisibility, it is true that x | a. Because divisibility is transitive and n|x and x|a, it is true that n | a.

QED

Finally,

Main Claim: A product of n consecutive natural numbers is divisible by n!.

Symbolically,

For all k, n in N, (n!) | [k(k+1)…(k+(n-1))]

Proof: By induction on n, let k and n be natural numbers.

As the base case, consider n = 1. It is true that 1! | k, or 1|k, for all k in the domain because 1 divides every number.

QED base case

As the inductive hypothesis, assume (n!) | [k(k+1)…(k+(n-1))]. Then it is sufficient to show that ((n+1)!) | [k(k+1)…(k+((n+1)-1))] or equivalently,

((n+1)!) | [k(k+1)…(k+n)]

We may also rearrange the terms such that:

(n+1)n!|k(k+1)… (k+n-1) (k+n)

To begin,

By the definition of divisibility,

[k(k+1)… (k+n-1) ]| [k(k+1)… (k+n-1) (k+n)]

Using our assumption that (n!) | [k(k+1)…(k+(n-1))], we conclude that

(n!)| [k(k+1)… (k+n-1) (k+n)] using the transitive property of divisibility.

Then we must also prove (n+1) | k(k+1)… (k+n-1) (k+n).

From lemma 3, we know that (n+1) | k(k+1)… (k+n-1) (k+n) is true because the product of (n+1) consecutive natural numbers is divisible by (n+1).

So now we know that (n+1)| k(k+1)… (k+n-1) (k+n) and (n!)| [k(k+1)… (k+n-1) (k+n)], so (n+1)n!|k(k+1)… (k+n-1) (k+n). By the definition of factorial, (n+1)n! =(n+1)! , so our original claim, ((n+1)!) | [k(k+1)…(k+n)], is true.

QED inductive step

QED claim ☺